International Journal of Theoretical Physics, Vol. 27, No. 5, 1988

Abstract Quantum Theory and Space–Time Structure. I. Ur Theory and Bekenstein–Hawking Entropy

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Received December 27, 1987

We discuss the close connection between a quantum theory of binary alternatives and the local Lorentzian structure of space-time, and outline v. Weizsäcker's concept of the "ur"—the quantized binary alternative. Then space-time is introduced mathematically as a symmetric space of the invariance group of the ur. It is physically interpreted as "the" cosmological space-time, the universe. In our model spacelike structures rest on the concept of "hypermembranes"—dynamical manifolds of codimension 1 in space-time. For a given number of urs a *smallest length* is introduced in this cosmic model by group-theoretic arguments. Already before introducing a dynamics the concept of isolated noncomposite objects can be given. They can be understood as simple models either for elementary particles or for black holes. Identifying the maximal localized states of many urs with a localized state of a particle, we get a good description of the large cosmological numbers and also a *lower bound for a neutrino mass.* A simple counting of the particle states given from the ur-theoretic ansatz allows an easy explanation of the Bekenstein-Hawking entropy.

One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seen to be in accordance with a continuum theory and must lead to an attempt to find a purely algebraic theory for the description of reality. But nobody knows how to obtain the basis of such a theory.—Albert Einstein, *The Meaning of Relativity*, 6th ed., Appendix IID.

1. INTRODUCTION

Quantum physics is extremely successful in its description of experience. This is a factum which has to be explained.

In his book, *Aufbau der Physik*, v. Weizsäcker (1985a) gives a survey of the foundations of physics and presents the idea that quantum theory should be taken as the basic theory for the whole physics.

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In a recent paper, Drieschner *et al.* (1987) give a detailed description of the reconstruction of abstract quantum theory. Only a very short and rough overview will be given here. "Abstract" quantum theory means quantum theory in Hilbert space without any reference to position space or to actually existing particles.

The starting point is to explain as far as possible quantum physics as a consequence of the preconditions of experience. In the context here, *experience means learning from the past to predict future events.* Therefore an elementary understanding of past and future is a precondition for every empirical science.

v. Weizsäcker's consideration of semantical consistency is in some sense an antiaxiomatic principle: We have to start with an elementary and imprecise understanding of conceptions which cannot be given in an axiomatic way. After we have developed the theory, we must take a new look at our starting concepts either to improve or to change them or to understand them in a better way.

Science examines decidable alternatives: Any object is defined by the alternatives that are decidable with respect to its states or properties. In any decision we want to get at least probabilities for the outcomes of the measurements. This implies a preunderstanding of probability and measuring. Drieschner defines probability as *the prediction of an expectation value for a relative frequency.* Such regressive definitions are also characteristic for quantum theory, e.g., multiple quantization can be understood as the application of the quantization procedure again to a quantum theory. This will be explained in a forthcoming paper.

2. RECONSTRUCTION OF ABSTRACT QUANTUM THEORY

2.1. The Impossibility of a Consistent Classical Theory

All stable basic objects in a classical theory, such as stable atoms in statistical thermodynamics or the stable rigid bodies in mechanics, cannot be explained without using quantum theory. As Planck already saw, it is a fact that a classical continuum cannot be thermodynamically stable without a (possibly hidden) quantum hypothesis. Therefore, we make the hypothesis:

Only quantum theory can be the basis for the whole of physics.

2.2. Reconstruction of Abstract Quantum Theory

To reconstruct quantum theory, v. Weizsäcker introduces three postulates:

(A) *Separable alternatives.* There exist separable, finite, empirically decidable alternatives.

(*B*) *Indeterminism.* With any pair of mutually exclusive states x, y in an alternative there exist states z with conditional symmetric probabilities different from 0 or 1 to find z for given x or y:

$$p(x, z) \neq (0 \text{ or } 1), \qquad p(y, z) \neq (0 \text{ or } 1)$$

(*C*) *Kinematics.* States of a given alternative develop in time in such a way that their relative probabilities remain unchanged.

From these postulates some consequences result:

(*i*) *State space.* The set of states for every n-fold alternative constitutes an n-dimensional vector space.

(*ii*) *Symmetry.* No state of an alternative is distinguished. There exists a probability-preserving symmetry group. The probabilities bring in the continuum, so the symmetry group will be a Lie group.

(*iii*) *Dynamics.* The states develop under the action of an one-dimensional subgroup of the symmetry group with time as its parameter.

(*iv*) *Preservation of states.* If a state is to be recognizable, there must exist a dynamics that keeps this state constant.

If a dynamics is to be observable, first it has to hold the alternative separated. Second, if it has no eigenstates, no state could be observed. So by this "Darwinistic" argument from the dynamics it follows:

(*v*) *Vector space.* The state space has to be a vector space over the complex numbers, moreover an n-dimensional Hilbert space.

Only the complex numbers are algebraically closed, so in this case we can always have diagonalizable self-adjoint generators for the possible dynamics.

(*vi*) *Composition.* Two alternatives are decided by deciding their Cartesian product. The state space of the product alternative is the tensor product of the state spaces of the two subalternatives.

The concept of "abstract" quantum theory can easily be recognized. No reference to a classical theory or to position space has been made.

3. THE DEFINITION OF THE Ur

It is the hope that by this theoretical concept (v. Weizsäcker 1955, 1971, 1985a,b; Scheibe *et al.*, 1958; Castell 1975) that the whole concrete quantum theory and with it the basic laws of physics can be derived by means of the additional postulate of urs:

(a) Every n-fold alternative can be decomposed into a product of binary alternatives.

(b) Every state space can be understood as a subspace of a tensor product of two-dimensional spaces.

After these two trivial-seeming statements we set up the central dynamical postulate:

(c) For any object there is at least one decomposition into binary (sub)objects—called urs—such that its dynamics is invariant under the symmetry group of the urs.

This postulate—all objects "arise from" or "consist of" urs—constitutes a radical abstract atomism. It is impossible to reduce science to something simpler than a set of binary alternatives. But we note explicitly that this conception has nothing to do with a setup of matter by *spatially* smallest objects.

So the ur is introduced as a (sub)object, quantum-theoretically described in a two-dimensional complex Hilbert space. The probability-preserving symmetry group for its states is built up from the $U(1)$ as the dynamical subgroup, from $SU(2)$ and the complex conjugation. This group was called the co-unitary group by Finkelstein *et al.* (1959).

4. THE INTRODUCTION OF POSITION SPACE

The close connection between a quantum theory of binary alternatives and the local Lorentzian structure of space–time has been known for a long time from the work of v. Weizsäcker (1955, 1958, 1971; Scheibe *et al.*, 1958) and, independently, of Finkelstein (1969). Whereas v. Weizsäcker looked for, so to speak, "energy-momentum alternatives," Finkelstein has done this for "space–time alternatives." So in some sense his view is conjugate to ours.

These old relations were rediscovered from another point of view by Noyes *et al.* (1987), who came from string theory. In this connection also the work of Chew and Stapp (1986) should be mentioned, who start from an electrodynamically oriented model.

In the present paper we will investigate first the global structure of space and second physical restrictions for the measurability of a smallest length.

In physics, position space is distinguished by several attributes: Every interaction between different objects depends on their relative position (in some cases also on the time derivatives of the position). Every observation happens by interaction; therefore, every measurement is first a measuring of position.

We call position space the space of *minimal* dimension in which the description of interacting objects is possible in general. This space is constructed by *projecting* all the individual configuration spaces of all objects into a single one of the same dimension. After this projection, all the positions of the different objects are in one and the same space. Therefore it becomes possible to speak of their spatial distances.

In a sample of interacting objects the strength of the interaction between them is not changed if the state of each one is transformed with the same group element from the group of motions in position space. On the other hand, if different objects are transformed by different group elements, then the strength of their interaction will be changed. This group-theoretic aspect shows that position space should be understood as a *symmetric space* of its group of motions.

For quantum physics the concept of position space is not a natural one. First, the localization of a single quantum object is a nontrivial problem. Second, two interacting objects can always be described as one object possessing internal degrees of freedom and, *a fortiori*, in almost all cases it is impossible to describe the whole object as a composite one, because the states in which the object is formed of its "parts" constitute only a set of measure zero in the whole state space. So one could formulate that the introduction of position space introduces the classical limit of *separated objects* and is in some sense in contradiction to quantum physics. An attempt to correct this drawback can be made by introducing *interaction.*

To introduce the concept of position space into the ur theory we use its group-theoretic aspect and the projection process described above. Such a projection process has been expressed more verbally as "space–time describes only a surface of reality" (v. Weizsäcker, 1985b).

If we project all the state spaces of the urs onto a single one, we can examine probability relations between states of different urs.

Since we propose all objects as consisting of urs, it is obvious that the probability relations between the objects should remain unchanged if we transforform all the urs by the same element from its symmetry group. But the probability relations will be changed if the urs constituting different objects are transformed by different group elements. We will interpret this phenomenon as a changing strength of interaction between those objects and we make the following hypotheses (v. Weizsäcker 1955, 1971):

(i) The parameter space for the strength of interaction is a symmetric space of the symmetry group of the ur.
(ii) The physical position space can be *identified* with this space.

If we make the idealization of the free ur, then its invariance group is essentially $U(2)$. The time development might be described by the subgroup

$U(1)$. So as symmetric space of the invariance group representing position space we have to choose

$$U(2)/U(1) = SU(2) = \mathbb{S}^3$$

In ur theory it is supposed that this is the reason for the three-dimensionality of position space.

5. COSMOGRAPHY

A typical structural element in the Ur-theoretic way of thought is the reversal of the usual order of the arguments. So we have to consider the spacelike structure of the cosmos prior to a description of gravitation and particles.

The states of an ur constitute an abstract $\mathbb{C}^2$. It is a representation space of a representation ${}^2D_{1/2}$ of $SU(2)$, which can be considered as a subrepresentation of the regular representation of this group in the Hilbert space

$$\mathbb{L}^2(SU(2)) = \mathbb{L}^2(\mathbb{S}^3)$$

The elements of $\mathbb{L}^2(\mathbb{S}^3)$ can be approximated by functions with smooth graphs of codimension 1 in a four-dimensional space–time. In analogy to hyperplanes (which are flat submanifolds of codimension 1), we will call them *hypermembranes.* Thus, we can represent the states of an ur by hypermembranes in the position space $\mathbb{S}^3$ possessing only one node plane. Hence an ur is an object extended over the whole cosmic space.

If there are more urs in this cosmos, they constitute higher dimensional representations of the symmetry group $SU(2)$. Each irreducible component again can be represented by sets of hypermembranes over $\mathbb{S}^3$ possessing different frequencies. In general there are multiplicities larger than one for these irreducible components. The multiplicities express the probabilities for the occurrence of the different irreducible representations. Consequently, if we have more urs, then it is possible to get a higher spatial resolution, because in higher dimensional representations sharper wave packets can be formed.[2]

[2]Before we investigate the spatial structure in this model we point out that no dynamics has been defined. That means that there is no assumption on the time development for the number of urs in this cosmic model and from the ur-theoretic concepts only the two "trivial" postulates are used. If we speak in the following about the "number of urs," we mean the expectation value of this expression, because at a fixed time the existence of a sharp number of uts cannot be supposed. Also, at this stage, the distinction mass and energy is not possible, since there is no description of motion.

Let R denote the radius of curvature of this cosmic space and let N be the number of urs in this cosmos at a given time. Then they constitute a representation $({}^2D_{1/2})^{\otimes N}$ of $SU(2)$. Its decomposition into irreducible components is given by (we define $|N/2| = k$ for $N = 2k$ *or* $N = 2k+1$)

$$({}^2D_{1/2})^{\otimes N} = \bigoplus_{j=0}^{|N/2|} \frac{N!(N+1-2j)}{(N+1-j)!j!}\, {}^{2|N/2|-2j+1}D_{|N/2|-j} \tag{1}$$

The wave functions of an irreducible representation ${}^nD$ have a wavelength of the order R/n (see, e.g., Vilenkin, 1968).[3]

To estimate the distribution of the wavelengths of the hypermembranes that can be found in a cosmos containing N urs, we have to investigate the factor of multiplicity

$$f(j) = \frac{N!(N+1-2j)}{(N+1-j)!j!} \tag{2}$$

The maximum for $f(j)$ is at[4] $j_{\max} = \frac{1}{2}[N - \sqrt{N+2}]$, so

$$j_{\max} \approx \tfrac{1}{2}(N - \sqrt{N}) \qquad \text{for} \quad N \gg 1 \tag{3}$$

For $f(j)$ we get, with

$$\ln(n!) \approx (n + \tfrac{1}{2}) \ln n - n + \tfrac{1}{2} \ln(2\pi)$$

the result

$$f(N/2) = O(2^{N/2} N^{-3/2})$$

$$f[\tfrac{1}{2}(N - \sqrt{N})] = O(2^{N/2} N^{-1})$$

$$f(0) = 1$$

After the maximum, we have between $j = \frac{1}{2}(N - \sqrt{N})$ and $j = 0$ an exponential decrease of $f(j)$ from the order of 2^N to 1. It can be approximated by

$$f(j_{\max} - \tfrac{1}{2}a) = 2^N \exp[-(1 + a/\sqrt{N})^2] \tag{4}$$

We see from this estimation that the multiplicities are large for representations ${}^kD$ with $0 \leq k \leq 2\sqrt{N}$. Above this point there is an exponential decrease for the multiplicities. Therefore the corresponding representations could be neglected.

The N urs constitute a highly decomposable representation of $SU(2)$ which mirrors the spatial structure in this cosmos. Without further assumptions we can say that representations with localizations sharper than $R/2\sqrt{N}$

[3]A basis in its representation space is given by the projections on the unit ball $\mathbb{S}^3 \subset \mathbb{C}^2$ of the homogeneous polynomials of degree $2n$ in two complex variables.

[4]It can be found by the ansatz $f(j_{\max}) = f(j_{\max} + 1)$.

almost never occur. So for a cosmos containing N urs we define a value of order $R/\sqrt{N}=\lambda_0$ as the shortest possible physically realizable length, which later will be identified with the Planck length.

6. COSMOLOGICAL ESTIMATES FOR PARTICLES

Now we have to look for states of urs constituting an isolated noncomposite object. We will start with the concept of a fixed time, so only the nondynamical aspects of the objects can be investigated.

In physics we know two types of "noncomposite" objects: some kinds of elementary particles and black holes. An elementary particle can be localized in space down to its Compton wavelength without disturbing its identity. A black hole is localized inside its Schwarzschild radius. Both of these conditions can be modeled in the ur concept.

Let n urs constitute a single object. Then the most localized state that possibly can be constructed from these urs has an extension of the order R/n. We suppose that such a state can be identified with a localized state of a particle and denote a volume of diameter R/n as an "elementary volume" for this kind of particle (localizing with respect to the three dimensions in space requires at most $3n_p$ urs and not n_p^3).

By our condition of a minimal length in the cosmic space this can be done only down to λ_0. This will create an upper bound for the number of urs constituting an elementary particle. To localize up to $R/N^{1/2}=\lambda_0$ we need $N^{1/2}$ urs for such a hypothetical object.

v. Weizsäcker first investigated the number of urs in our universe. He referred to the fact that ponderable matter is built up in its essence by nucleons and that observations give a radius of $10^{40}\lambda_{\text{proton}}$ for the metagalaxies. He claimed that the number of urs should be equal to the number of elementary proton volumes $\lambda^3_{\text{proton}}$ in the metagalaxies. This means that the number of decidable questions can be expressed by the number of bits deciding for each such volume to be either empty of filled up.

In a fundamental theory this is a natural condition for a first guess. Because there is no more fundamental proposition, we can check our ideas only by *consistency considerations.* Conceptual thinking cannot represent the wholeness of reality, so any physical theory has a limited force of explanation (Drieschner *et al.*, 1987; Görnitz and v. Weizsäcker, 1987). We want to define separate alternatives. In a first step it seems meaningful to identify the statements about separate alternatives with the statements about the possibilities for the distribution of the fundamental parts of matter in space, *i.e., we use these as a representation for the set of meaningful decidable questions.* All measuring devices, all clocks and rods, are made from ponderable matter. Since space is essentially void, the approximation of *separate* alternatives is in this case an adequate one. The postulate was

$$N = R^3/\lambda_p^3 \tag{5a}$$

With

$$\lambda_p = 10^{-40}R \tag{5b}$$

we get

$$N = R^3/(10^{-40}R)^3 \qquad \text{or} \qquad N = 10^{120} \tag{5c}$$

So he got for the number of urs the value $N = 10^{120}$ and then he had to conclude that about 10^{80} nucleons should exist in the universe if all the urs formed nucleons. The resulting density for matter of 10^{-1} proton/cm^3 is of the order of the observed value.

There also can be a lower bound of the ur number for a *separated* elementary particle. It will be reached when the whole cosmic space is filled up with occupied elementary volumes of those objects. Let n_n urs constitute such a "neutrino"; then their maximal number in the universe is N/n_n. The diameter of the elementary volume is $\lambda_n = R/n_n$. From

$$R^3 = (N/n_n)\lambda_n^3 \tag{6a}$$

we get

$$R^3 = (N/n_n)(R^3/n_n^3) \qquad \text{or} \qquad n_n = N^{1/4} \tag{6b}$$

such that, if all the urs formed neutrinos, there would be atmost $N^{3/4}$ of these particles.

Taking for the number of urs $N = 10^{120}$, we get for the minimal length the value $l_0 = R/10^{60}$. If we compare this with the length λ_{proton}, we see that λ_0 is of the order of the Planck length.

The condition that each particle possesses its own elementary volume can be seen in analogy to the features of fermions.[5]

Our estimates remain valid also if the urs are roughly equally distributed over a few kinds of particles (see also Section 7). If this special idea can be further confirmed, it would result in a rest mass for neutrinos of an order smaller than $10^{-10}m_{\text{proton}}$. Neutrinos with a substantially smaller rest mass, if there are any, can then be formed only from a negligible part of all the urs in the universe and are unable to contribute to its total mass in a substantial way.

For particles of the bosonic type the idea of their "own" elementary volume is very dubious. The wavelength for these particles can be as large as the radius of the universe (but in such a case it seems hard to speak

[5]Hence we have called them "neutrinos" and "protons"; at this stage a distinction between electrons and protons is not yet possible.

about "particles"). So we request that in the average the urs be in states with such a wavelength that the volume of the cosmic space is equal to the cube of this wavelength times the number of all these extended objects. Then in the average every such small volume is occupied by only one such "boson." Since in this situation there is nothing to decide, it looks like a thermal equilibrium and we will call these objects "photons." Due to our conditions we get

$$R^3 = N\lambda_{\rm ph}^3 \tag{7a}$$

and it follows that

$$R^3 = (N/n_{\rm ph})(R^3/n_{\rm ph}^3) \qquad \text{or} \qquad n_{\rm ph} = N^{1/4} \tag{7b}$$

such that $N^{3/4} = 10^{90}$ of these "photons" could exist. The relation between the maximal numbers of photons and of nucleons (which also remains valid if instead of N urs only parts between $10^{-0}N$ and $10^{-2}N$ are forming these particles) is

$$z_{\rm photon}/z_{\rm nucleon} = 10^{10} \tag{8}$$

which is of the order of the empirical value.

If we identify the energy of a particle with an extension of the order of the Planck length with the Planck energy of 10^{19} GeV, then our model "nucleon" has a mass of order 0.1 GeV and the temperature of the "photons" is of the order 10^2 K. A single ur corresponds then to an energy quantum of 10^{-32} eV.

So the ur theory can offer a natural way to explain the so-called large cosmological numbers.

7. THEORY AND THE BEKENSTEIN–HAWKING ENTROPY

Entropy means missing information—an ur represents a bit of information—so there has to be a close relation between these two conceptions. As v. Weizsäcker (1971, Chapter III5) has mentioned, information is related to a concept, a basic idea. In physics normally entropy is related to the *particle concept* (respectively, to a quantum field) and not to the ur concept. If we want to do the same here, we have to look for possible (virtual) particle states inside the objects under consideration. If we take as microstates the sense of Boltzmann the distribution of particles in space, i.e., the occupation number of elementary volumes by particles, we can estimate a particle-related entropy.

By an *informatively closed* or *irreducible* volume we understand a finite volume $V \approx R^3$ on which no information about its internal states can be obtained from "outside." This can happen because the volume is itself a closed space or by a horizon.

We will start with a closed cosmic space with a curvature radius R and use Planck-Wheeler units. The basic objects—the urs—are represented by the hypermembranes of wavelength R and energy $1/R$. Let N be the total number of urs; then $N = R^2$. The total number of mutually orthogonal states of all urs is 2^N, so the entropy with respect to the urs is of order N. The energy of one ur is $1/R$, so for N urs it is R and we get

$$\text{entropy} \approx (\text{energy})^2 \tag{9}$$

Now we compute an entropy related to localized particles. Let n_i urs with $n_i \leq \sqrt{N}$ form a particle of mass-energy $m_i = n_i/R$ and elementary volume $v_i = \lambda_i^3 = (R/n_i)^3$. Then the number z_i of the i particles can be as large as $z_i = N/n_i$, and the number of places P_i is equal to

$$P_i = V/v_i = R^3/(R/n_i)^3 = n_i^3$$

The thermodynamic probability of putting z_i particles of a bosonic type on P_i places is

$$W_{Bi} = (P_i + z_i)!/\{P_i!z_i!)\tag{10}$$

For a general n_i we get

$$\begin{aligned} S_{Bi} &= \ln W_{Bi} \\ &= (n_i^3 + N/n_i)\ln(n_i^3 + N/n_i) - n_i^3 \ln n_i^3 - N/n_i \ln(N/n_i) \\ &= n_i^3[\ln(1 + N/n_i^4) - N/n_i^4 \ln(1 + n_i^4/N)] \end{aligned}\tag{11a}$$

With $\alpha_i = n_i^4/N$, $N^{-1} \leq \alpha_i \leq N$, we get

$$S_{Bi} = N^{3/4}\alpha_i^{3/4}[\ln(1+\alpha_i^{-1}) - \alpha_i^{-1}\ln(1+\alpha_i)] = N^{3/4}f_B(\alpha_i)\tag{11b}$$

The entropy S_{Bi} has its maximal value at $\alpha = 17.5$ or $n_i \approx 2N^{-1/4}$. With $f_B(17.5) = 1.902$ we get

$$S_{B\max} \approx 2N^{3/4}\tag{11c}$$

If at most one particle can be at a place, then we will denote these objects as fermionic particles. The thermodynamic probability to put z_i fermionic particles on P_i places is

$$W_{Fi} = (P_i)!/\{(P_i - z_i)!(z_i)!\}\tag{12}$$

For $n_i = N^{1/4}$ it is $W_{Fi} = 1$,[6] and for $n_i = N^{1/2}$ we have

$$\begin{aligned} \ln W_{Fi} &= N^{3/2}\ln(N^{3/2}) - (N^{3/2} - N^{1/2})\ln(N^{3/2} - N^{1/2}) - N^{1/2}\ln N^{1/2} \\ &\approx N^{1/2}(1 + \ln N) \end{aligned}\tag{13}$$

[6]This was the reason for the lower bound for the fermion rest mass.

For a general n_i we get

$$S_{Fi} = \ln W_{Fi}$$
$$= n_i^3 \ln n_i^3 - (n_i^3 - N/n_i) \ln(n_i^3 - N/n_i) - N/n_i \ln(N/n_i) \qquad (14)$$

With $n_i^4 = \alpha_i N$, $1 < \alpha_i \leq N$, one has

$$S_{Fi} = N^{3/4} \alpha_i^{3/4} [-\ln(1 - \alpha_i^{-1}) + \alpha_i^{-1} \ln(\alpha_i - 1)] = N^{3/4} f_F(\alpha_i) \qquad (15a)$$

The entropy S_{Fi} has its maximal value at $\alpha = 22.5$ or $n_i \approx 2N^{-1/4}$; with $f_F(22.5) = 1.8782$ we get

$$S_{\max Fi} \approx 2N^{3/4} \qquad (15b)$$

The maximal entropy $S_{\max}$ in the Bose as well in the Fermi case is equal to the value given by the simple calculations in Section 6.

The calculations above are made under the extreme condition that all urs form only a single kind of particle. The other extreme is the assumption that all possible particles of all possible numbers of urs in it will be present. In this case the total entropy S_i for localized particles can be computed by means of the partial thermodynamic probabilities for n_i from 1 to $N^{1/2}$. Let z_{Bi} and z_{Fi} be the number of particles (bosons or fermions) made from n_i urs. Then

$$S_i = \ln \sum_{z_i} \prod_1^{N^{1/2}} W_i = \ln \sum_{z_i} \prod_1^{N^{1/2}} \frac{(P_i + z_{Bi})! \, P_I!}{P_i! \, z_{Bi}! \, (P_i - z_{Fi})! \, z_{Fi}!} \qquad (16a)$$

$$S_i = \ln \sum_{z_i} \prod_1^{N^{1/2}} \frac{(n_i^3 + z_{Bi})}{z_{Bi}! \, (n_i^3 - z_{Fi})! \, z_{Fi}!} \qquad (16b)$$

Thereby the sum over z_i must be taken over all combinations for all z_{Bi} and z_{Fi} with $0 \leq z_{Bi} + z_{Fi} \leq N/n_i$ under the secondary condition

$$\sum n_i (z_{Bi} + z_{Fi}) = N \qquad (17)$$

As usual in thermodynamics, S_i can be approximated by the logarithm of the largest term of the sum (16), which is given if the z_{Bi} and the z_{Fi} have the values

$$z_{Bi0} = n_i^3 / [\exp(\mu n_i) - 1] \qquad (18a)$$

$$z_{Fi0} = n_i^3 / [\exp(\mu n_i) + 1] \qquad (18b)$$

μ is a Lagrange parameter, which can be computed from

$$N = \sum_{n_i=1}^{N^{1/2}} n_i (z_{Bi0} + z_{Fi0})$$
$$= \sum_{n_i=1}^{N^{1/2}} n_i^4 \{[\exp(\mu n_i) - 1]^{-1} + [\exp(\mu n_i) + 1]^{-1}\} \qquad (19)$$

An approximation to equation (19) can be found by

$$N=\sum_{n_i=1}^{n_i=N^{1/2}}\frac{n_i^4}{\exp(\mu n_i)-1}+\frac{n_i^4}{\exp(\mu n_i)+1}$$
$$\approx\int_{x=1}^{x=N^{1/2}}\left(\frac{x^4}{\exp(\mu x)-1}+\frac{x^4}{\exp(\mu x)+1}\right)dx$$
$$=\mu^{-5}\int_{y=\mu}^{y=\mu N^{1/2}}\left(\frac{y^4}{e^y-1}+\frac{y^4}{e^y+1}\right)dy$$
$$\approx\mu^{-5}\int_{x=0}^{x=\infty}\left(\frac{x^4}{e^x-1}+\frac{x^4}{e^x+1}\right)dx \tag{20}$$

which gives (see Gradstein and Ryshik (1963, 3.411.1 and 3)

$$N\approx 2\mu^{-5}\Gamma(5)\zeta(5)\approx 240\cdot 1.04\mu^{-5} \tag{21}$$

or

$$\mu\approx 3.0N^{-1/5}\approx N^{-1/5} \tag{21'}$$

For the total entropy S_t we get

$$S_t=\sum_{n=1}^{N^{1/2}}\ln\frac{[n^3+n^3/(e^{\mu n}-1)]!}{[n^3/(e^{\mu n}-1)]![n^3-n^3/(e^{\mu n}+1)]![n^3/(e^{\mu n}+1)]!} \tag{22a}$$

$$S_t=\sum_{n=1}^{N^{1/2}}n^3\left(\frac{\mu n}{e^{\mu n}-1}+\frac{\mu n}{e^{\mu n}+1}+\ln\frac{e^{\mu n}+1}{e^{\mu n}-1}\right) \tag{22b}$$

which by (19) is equal to

$$S_t=\mu N+\sum\left(n^3\ln\frac{\mu n+1}{e^{\mu n}-1}\right) \tag{22c}$$

This can be approximated by

$$S_t\approx\mu N+\int_{x=1}^{x=N^{1/2}}x^3\ln\frac{e^{\mu x}+1}{e^{\mu x}-1}\,dx\approx\mu N+\mu^{-4}\int_{x=0}^{x=\infty}x^3\ln\frac{e^x+1}{e^x-1}\,dx \tag{23}$$

For the integral we get by numerical integration a value of order 12.05, so

$$S_t\approx\mu N+12\mu^{-4}\approx 3N^{4/5}+12(3N^{-1/5})^{-4}\approx N^{4/5} \tag{24}$$

This value for S_t gives an upper limit for the entropy of localized particles. In reality we have only few kinds of particles, therefore the estimates (11c) and (15b) give a better approach to the empirical value, which is of order 10^{90}. For the value $N=10^{120}$ this corresponds to $N^{3/4}$.

The picture of a localized particle used above already has many classical aspects. It is not so bad for thermal photons with their limited coherence length, but in general a quantum particle has many nonlocalized states.

If a particle is built up from n_i urs, then it should have about $\exp(n_i \ln 2) - P_i$ nonlocalized quantum states and the thermodynamic probability (16a) should get an additional factor:

$$W_{i(\text{nonlocal})} = [\exp(n_i \ln 2) - P_i] W_i \tag{16a$'$}$$

and the total "nonlocal" entropy $S_{t\text{nl}}$ gets an additional term:

$$\begin{aligned} S_{t\text{nl}} &\approx S_i + \sum (n \ln 2 - 3 \ln n) \\ S_{t\text{nl}} &\approx N^{4/5} + [N^{1/2}(N^{1/2}+1)/2] \ln 2 \end{aligned} \tag{24$'$}$$

so that

$$S_{t\text{nl}} \approx \tfrac{1}{2} N \ln 2 \approx N \tag{25}$$

This is indeed of order of the total number of urs. In the sense of semantical consistency this result can be interpreted that the urs give the information of *all* possible *quantum* states for the particles.

After these considerations, for a cosmic space we will look at *irreducible volumes inside a cosmos.* Let n_{bh} urs with $n_{\text{bh}} > \sqrt{N}$ form a single object. Then $z_{\text{bh}} = N/n_{\text{bh}}$ of them could exist at most. We will call them irreducible if their maximal extension λ_{bh} is equal to R/z_{bh}:

$$\lambda_{\text{bh}} = R/z_{\text{bh}} \tag{26}$$

This simple condition mimics much of the properties of a Schwarzschild horizon. We get

$$\lambda_{\text{bh}} = R/z_{\text{bh}} = R(n_{\text{bh}}/N) = n_{\text{bh}}/R \tag{27}$$

The mass-energy of an ur is $1/R$, so the mass of an object of n_{bh} urs will be

$$m_{\text{bh}} = n_{\text{bh}}(1/R) = n_{\text{bh}}/R \tag{28}$$

For these irreducible objects mass-energy and linear extension are of the same order.

The fundamental hypermembranes for an irreducible object with spatial extension λ_{bh} have an extension of order λ_{bh} and are formed by σ_{bh} urs with

$$\lambda_{\text{bh}} = R/\sigma_{\text{bh}} \tag{29}$$

Therefore

$$\sigma_{\text{bh}} = R^2/n_{\text{bh}} \tag{30}$$

The number of these hypermembranes N_{bh} is

$$N_{\text{bh}} = n_{\text{bh}}/\sigma_{\text{bh}} = n_{\text{bh}}^2/R^2 = m_{\text{bh}}^2 \tag{31}$$

Now it is possible to make the computations with these hypermembranes in the case of an irreducible object in the same manner as with the urs in the case of a cosmos. Therefore we get for the entropy of the internal states of such an irreducible object the value

$$S_{\mathrm{bh}} \approx m_{\mathrm{bh}}^2 \tag{32}$$

This is, up to a constant factor of order one, the Bekenstein–Hawking entropy for a Schwarzschild black hole. If it is permitted to identify the irreducible objects with Schwarzschild black holes, then the ur-theoretic model gives a very simple explanation for its entropy. This entropy is then to be interpreted as an expression for the missing knowledge about its internal degrees of freedom. These degrees of freedom are described by the relation between the "constituting urs" of the object and the information about its spatial extension inside a spatially closed cosmic space.

ACKNOWLEDGMENTS

I am grateful to the Deutsche Forschungsgemeinschaft for financial support. I thank Prof. C. F. v. Weizsäcker for his continuous support and for countless helpful discussions. I thank also Prof. H.-D. Doebner as well as Prof D. Finkelstein and Dr. S. Finkelstein for their hospitality and stimulating interest and Dr. E. Ruhnau for critical remarks on the manuscript.

REFERENCES

Castell, L. (1975). Quantum theory of simple alternatives, in *Quantum Theory and the Structure of Time and Space*, Vol. II, Hanser, München.

Chew, G. F., and Stapp, H. P. (1986). Pregeometrical quantum mechanics, Lawrence Berkeley Laboratory preprint.

Drieschner, M. (1970). Quantum mechanics as a general theory of objective prediction, Thesis, Hamburg.

Drieschner, M. (1979). *Voraussage, Wahrscheinlichkeit, Objekt*, Lecture Notes in Physics 99, Springer, Berlin.

Drieschner, M., Görnitz, Th., and Weizsäcker, C. F. v. (1987). Reconstruction of abstract quantum theory, *International Journal of Theoretical Physics.*

Finkelstein, D. (1969). Space-time code, *Physical Review*, **184**, 1261.

Finkelstein, D., Jauch, J. M., and Speiser, D. (1959). CERN Report 59-7.

Gradstein, I. S., and Ryshik, I. M. (1963). *Tables of Integrals, Sums, Series and Products*, Fis-Mat-Lit, Moscow.

Görnitz, Th., and Weizsäcker, C. F. v. (1987). Quantum interpretations, *International Journal of Theoretical Physics*, **26**, 921.

Noyes, H. P., McGoveran, D. O., Manthey, J. M., Etter, T., Kilmister, C. W., Bastin, T., Gefwert, C., and Stein, I. (1987). A discrete relativistic quantum physics, SLAC-PUB 4347.

Scheibe, E., Süssmann, G., and Weizsäcker, C. F. v. (1958), Mehrfache Quantelung, Komplementarität und Logik III, *Zeitschrift für Naturforschung*, **13a**, 705.

Vilenkin, N. Ja. (1968). *Special Functions and the Theory of Group Representations*, American Mathematical Society, Providence, Rhode Island.

Weizsäcker, C. F. v. (1955). Komplementarität und Logik I, *Naturwissenschaften*, **42**, 521-529, 545-555.

Weizsäcker, C. F. v. (1958). Komplementarität und Logik II, *Zeitschrift für Naturforschung*, **13a**, 245.

Weizsäcker, C. F. v. (1971). *Die Einheit der Natur*, Hanser, Munich [*The Unity of Nature*, Farrar, Straus, Giroux, New York, 1980].

Weizsäcker, C. F. v. (1985a). *Aufbau der Physik*, Hanser, Munich.

Weizsäcker, C. F. v. (1985b). Quantum theory and space-time, in *Symposium on the Foundations of Modern physics, Joensuu*, P. Lathi and P. Mittelstaedt, eds.